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Deterministic and random coincidence point results for f-nonexpansive maps

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Abstract

Some deterministic and random coincidence theorems for f-nonexpansive maps are obtained. As applications, invariant approximation theorems are derived. Our results unify, extend and complement various known results existing in the literature.
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Keywords: Coincidence point; Random coincidence point; Random operator; Invariant approximation; Banach space

1. Introduction and preliminaries

Let M be a subset of a normed space X. We denote by $CD(M)$, $CB(M)$, and $K(M)$ the families of all nonempty closed, nonempty closed bounded, and nonempty compact subsets of M, respectively. The Hausdorff metric induced by d on $CD(M)$ is given by

$$H(A, B) = \max\Big\{\sup_{a\in A} \operatorname{dist}(a, B),\ \sup_{b\in B} \operatorname{dist}(b, A)\Big\}$$

for $A, B \in CD(M)$, where $\operatorname{dist}(a, B) = \inf_{b\in B} d(a, b)$. The set $P_M(u) = \{x \in M\colon\ d(x, u) = \operatorname{dist}(u, M)\}$ is called the set of best approximants to $u \in X$ out of M. Let $f : M \to M$. A mapping $T : M \to CD(M)$ is called f-Lipschitz if there exists $k \geqslant 0$ such that $H(Tx, Ty) \leqslant k\|fx - fy\|$ for any $x, y \in M$. If $0 \leqslant k < 1$ (respectively $k = 1$), then T is called an f-contraction (respectively f-nonexpansive map). A point $x \in M$ is called a coincidence point (respectively common

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fixed point) of f and T if $fx \in Tx$ (respectively $x = fx \in Tx$). The set of coincidence points of f and T is denoted by $C(f,T)$. The set of fixed points of T (respectively f) is represented by $F(T)$ (respectively $F(f)$). The pair $\{f,T\}$ is called (1) commuting if $Tfx = fTx$ for all $x \in M$ and (2) weakly compatible [5] if f and T commute at their coincidence points. The mapping f is called T-weakly commuting [8] if for all $x \in M$, $ffx \in Tfx$. If the pair $\{f,T\}$ is weakly compatible, then f is T-weakly commuting at the coincidence points. However, the converse is not true in general. If T is single-valued, then T-weak commutativity at the coincidence points is equivalent to the weak compatibility (see [8]). The mappings f and T are said to satisfy property $(E.A)$ [8] if there exist a sequence $\{x_n\}$ in X, some $a \in X$ and $A \in CD(X)$ such that $\lim_{n\to\infty} fx_n = a \in A = \lim_{n\to\infty} Tx_n$. The set M is called q-starshaped with $q \in M$ if the segment $[q,x] = \{(1-k)q + kx\colon 0 \leqslant k \leqslant 1\}$ is contained in M for all $x \in M$.

A Banach space X satisfies Opial's condition if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \neq x$. The map $T: M \to CD(X)$ is said to be demiclosed at 0 if for every sequence $\{x_n\}$ in M and $\{y_n\}$ in X with $y_n \in Tx_n$ such that $\{x_n\}$ converging weakly to x and $\{y_n\}$ converges to $0 \in X$, then $0 \in Tx$. A mapping $T: M \to CD(X)$ is said to satisfy condition (A) [14] if for any sequence $\{x_n\}$ in M, $D \in CD(M)$ such that $\operatorname{dist}(x_n, D) \to 0$ and $\operatorname{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$, then there exists $y \in D$ with $y \in Ty$. Let $f: M \to X$ be a mapping. Then f and T are said to satisfy condition (A^0) [13] if for any sequence $\{x_n\}$ in M, $D \in CD(M)$ such that $\operatorname{dist}(x_n, D) \to 0$ and $\operatorname{dist}(fx_n, Tx_n) \to 0$ as $n \to \infty$, there exists $y \in D$ with $fy \in Ty$.

Let (Ω, Σ) be a measurable space. A mapping $T: \Omega \to CB(M)$ is called measurable if for any open subset C of M,

$$T^{-1}(C) = \{\omega \in \Omega\colon T(\omega) \cap C \neq \emptyset\} \in \Sigma.$$

A mapping $\xi: \Omega \to M$ is said to be a measurable selector of a measurable mapping $T: \Omega \to CB(M)$ if ξ is measurable and for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega)$. A mapping $T: \Omega \times M \to CB(M)$ (respectively $f: \Omega \times M \to M$) is called a random operator if for any $x \in M$, $T(.,x)$ (respectively $f(.,x)$) is measurable. A measurable mapping $\xi: \Omega \to M$ is called a random fixed point of a random operator $T: \Omega \times M \to CB(M)$ (respectively $f: \Omega \times M \to M$) if for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))$ (respectively $f(\omega, \xi(\omega)) = \xi(\omega)$). A measurable mapping $\xi: \Omega \to M$ is a random coincidence point of random operators $T: \Omega \times M \to CB(M)$ and $f: \Omega \times M \to M$ if for every $\omega \in \Omega$, $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$. A random operator $T: \Omega \times M \to CB(M)$ (respectively $f: \Omega \times M \to M$) is said to be continuous (weakly continuous, nonexpansive etc.) if for each $\omega \in \Omega$, $T(\omega, .)$ (respectively $f(\omega, .)$) is continuous (weakly continuous, nonexpansive, etc.).

Latif and Tweddle [11] established some coincidence point theorems for f-nonexpansive mappings using the commutativity condition of maps. Afterwards, Shahzad and Latif [15] obtained random versions of their results. Recently, Shahzad [13] proved some general random coincidence point theorems and, as applications, derived a number of random fixed point results. In this paper, we obtain some coincidence point results. We note that the assumption of commutativity of maps in Latif and Tweddle's theorems and their random analogues are superfluous. We further add that we do not require f and T to be continuous in our main deterministic results. We apply our results to prove some fixed point theorems for a more general class of noncommuting maps. As applications, invariant approximation results are derived. Finally, we obtain random versions of our results using a general random coincidence point result due to Shahzad [13]. Our results unify, extend and complement many known results existing in the literature including those of Beg and Shahzad [1–3], Dotson [4], Jungck and Sessa [6], Jungck [7], Kamran [8], Latif and Bano [10], Latif and Tweddle [11], Shahzad [13], Shahzad and Latif [15], Tan and Yaun [17] and Xu [18].

The following results will be needed.

Theorem 1.1. [16] *Let (X,d) be a metric space, $f:X\to X$ and $T:X\to CD(X)$ such that $T(X)\subset f(X)$. If $f(X)$ is complete and T is an f-contraction, then $C(f,T)\neq\emptyset$.*

Lemma 1.2. [11] *Let M be a nonempty weakly compact subset of a Banach space X satisfying Opial's condition. Let $f:M\to M$ be a weakly continuous mapping and $T:M\to K(M)$ an f-nonexpansive map. Then $(f-T)$ is demiclosed.*

Theorem 1.3. [13] *Let M be a nonempty separable weakly compact subset of a Banach space X and $f:\Omega\times M\to M$ a random operator which is both continuous and weakly continuous. Assume that $T:\Omega\times M\to CB(M)$ is a continuous random operator such that $(f-T)(\omega,.)$ is demiclosed at 0 for each $\omega\in\Omega$. If f and T have a deterministic coincidence point, then f and T have a random coincidence point.*

Theorem 1.4. [13] *Let M be a nonempty separable complete subset of a metric space X, and $f:\Omega\times M\to M$ and $T:\Omega\times M\to CD(M)$ continuous random operators satisfying condition (A^0). If f and T have a deterministic coincidence point, then f and T have a random coincidence point.*

2. Coincidence point results

We begin with the following result, which extends and improves Theorem 2.1 of Latif and Tweddle [11].

Theorem 2.1. *Let M be a nonempty complete and q-starshaped subset of a normed space X and $f:M\to M$ a map such that $f(M)=M$. Assume that $T:M\to CD(M)$ is an f-nonexpansive map. If $T(M)$ is bounded and $(f-T)(M)$ is closed, then $C(f,T)\neq\emptyset$.*

Proof. Choose a sequence $\{k_n\}$ with $0<k_n<1$ such that $k_n\to 1$ as $n\to\infty$. For each n, define T_n by

$$T_nx=(1-k_n)q+k_nTx$$

for all $x\in M$. Then, for each n, $T_n:M\to CD(M)$, $T_n(M)\subset M=f(M)$, and

$$H(T_nx,T_ny)=k_nH(Tx,Ty)\leqslant k_n\|fx-fy\|$$

for each $x,y\in M$. Since $f(M)$ is complete, by Theorem 1.1, for each n, there exists $x_n\in M$ such that $fx_n\in T_nx_n$. This implies that $fx_n-y_n=(1-k_n)(q-y_n)$ for some $y_n\in Tx_n$. Since $T(M)$ is bounded and $k_n\to 1$ as $n\to\infty$, it follows that $fx_n-y_n\to 0$ as $n\to\infty$. Since $(f-T)(M)$ is closed, it follows that $0\in(f-T)(M)$ and so $fx_0\in Tx_0$ for some $x_0\in M$. Hence $C(f,T)\neq\emptyset$. □

The following result generalizes and improves Theorem 2.2 of Latif and Tweddle [11], Theorem 3.2 of Lami Dozo [9], and Corollary 3.4 of Jungck [7].

Theorem 2.2. *Let M be a nonempty weakly compact and q-starshaped subset of a Banach space X and $f : M \to M$ be a map such that $f(M) = M$. Assume that $T : M \to CD(M)$ is an f-nonexpansive map and that one of the following two conditions is satisfied*:

(a) $(f - T)$ *is demiclosed at* 0;
(b) f *is weakly continuous, T is compact-valued and X satisfies Opial's condition.*

Then $C(f, T) \neq \emptyset$.

Proof. As in the proof of Theorem 2.1, $fx_n - y_n \to 0$ as $n \to \infty$ where $y_n \in Tx_n$. By the weak compactness of M, there is a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ such that $\{x_m\}$ converges weakly to $y \in M$ as $m \to \infty$.

(a) Since $(f - T)$ is demiclosed at 0, we have $0 \in (f - T)y$. Thus $C(f, T) \neq \emptyset$.

(b) By Lemma 1.2, $(f - T)$ is demiclosed at 0. Hence the result follows from (a). □

Example 2.3. Let $X = \mathbf{R}$ with the usual norm and $M = [0, 1]$. Define

$$Tx = [0, x^2] \quad \text{and} \quad fx = 1 - x^2$$

for $x \in M$. Then all hypotheses of Theorems 2.1 and 2.2 are satisfied. Note that $x = 1/\sqrt{2}$ is a coincidence point of f and T. Note also that Theorems 2.1 and 2.2 of Latif and Tweddle [11] cannot be used here since f and T are not commuting.

The following extends Theorem 2.3 of Latif and Tweddle [11], Corollary 3.2 of Jungck [7], and Theorem 1 of Dotson [4].

Theorem 2.4. *Let M be a nonempty complete and q-starshaped subset of a normed space X, and $f : M \to M$ a map such that $f(M) = M$. Assume that $T : M \to CD(M)$ is an f-nonexpansive map. If f and T satisfy condition (A^0) and $T(M)$ is bounded, then $C(f, T) \neq \emptyset$.*

Proof. As in the proof of Theorem 2.1, $fx_n - y_n \to 0$ as $n \to \infty$ where $y_n \in Tx_n$. Since $\operatorname{dist}(fx_n, Tx_n) \to 0$ as $n \to \infty$, by condition (A^0) there exists an $x_0 \in M$ such that $fx_0 \in Tx_0$. □

Corollary 2.5. *Let M be a nonempty compact and q-starshaped subset of a normed space X, and $f : M \to M$ a continuous map such that $f(M) = M$. Assume that $T : M \to CD(M)$ is an f-nonexpansive map. Then $C(f, T) \neq \emptyset$.*

Theorem 2.6. *Let M be a nonempty complete and q-starshaped subset of a normed space X and $f : M \to M$ a map such that $f(M) = M$. Assume that $T : M \to CD(M)$ is an f-nonexpansive map, $T(M)$ is bounded, and $(f - T)(M)$ is closed. If, in addition, f is T-weakly commuting at v and $ffv = fv$ for $v \in C(f, T)$, then $F(f) \cap F(T) \neq \emptyset$.*

Proof. By Theorem 2.1, $C(f, T) \neq \emptyset$. Suppose $v \in C(f, T)$. Then $fv = ffv \in Tfv$. Hence $F(f) \cap F(T) \neq \emptyset$. □

The following result extends Theorem 6 of Jungck and Sessa [6].

Theorem 2.7. *Let M be a nonempty weakly compact and q-starshaped subset of a Banach space X and $f:M\to M$ be a map such that $f(M)=M$. Assume that $T:M\to CD(M)$ is an f-nonexpansive map and that one of the following two conditions is satisfied*:

(a) *$(f-T)$ is demiclosed at* 0;
(b) *f is weakly continuous, T is compact-valued and X satisfies Opial's condition.*

If f is T-weakly commuting and $ffv=fv$ for $v\in C(f,T)$, then $F(f)\cap F(T)\neq\emptyset$.

Proof. By Theorem 2.2, in both cases, $C(f,T)\neq\emptyset$. As in the proof of Theorem 2.6, $F(f)\cap F(T)\neq\emptyset$. □

Theorem 2.8. *Let M be a nonempty complete and q-starshaped subset of a normed space X, and $f:M\to M$ a map such that $f(M)=M$. Assume that $T:M\to CD(M)$ is an f-nonexpansive map, f and T satisfy condition (A^0), and $T(M)$ is bounded. If f is T-weakly commuting and $ffv=fv$ for $v\in C(f,T)$, then $F(f)\cap F(T)\neq\emptyset$.*

Proof. By Theorem 2.4, $C(f,T)\neq\emptyset$. As in the proof of Theorem 2.6, $F(f)\cap F(T)\neq\emptyset$. □

Next we derive some invariant approximation results.

Theorem 2.9. *Let M be a subset of a normed space X, $u\in X$, $f:X\to X$ and $T:X\to CD(X)$. Assume that $P_M(u)$ is nonempty q-starshaped and complete, $f(P_M(u))=P_M(u)$, T is f-nonexpansive on $P_M(u)$, $P_M(u)$ is T-invariant, and that one of the following two conditions is satisfied*:

(a) *$(f-T)(P_M(u))$ is closed*;
(b) *f and T satisfy condition (A^0).*

Then $P_M(u)\cap C(f,T)\neq\emptyset$. If, in addition, f is T-weakly commuting and $ffv=fv$ for $v\in C(f,T)$, then $P_M(u)\cap F(f)\cap F(T)\neq\emptyset$.

Proof. Since $T(P_M(u))\subset P_M(u)$, it follows that $T:P_M(u)\to CD(P_M(u))$. If (a) holds, then the result follows from Theorem 2.6. If (b) holds, the results follows from Theorem 2.8. □

The following corollary extends and improves Theorem 3.14 of Kamran [8]. We further note that Kamran's result remains true if the following assumption is dropped:

f and A_λ satisfy the property $(E.A)$ for each $\lambda\in[0,1]$ where $A_\lambda(x)=(1-\lambda)q+\lambda Tx$.

Corollary 2.10. *Let M be a subset of a normed space X, $u\in X$, $f:X\to X$ and $T:X\to CD(X)$. Assume that $P_M(u)$ is nonempty q-starshaped and compact, $f(P_M(u))=P_M(u)$, T is f-nonexpansive on $P_M(u)$, $P_M(u)$ is T-invariant, and f is continuous on $P_M(u)$. Then $P_M(u)\cap C(f,T)\neq\emptyset$. If, in addition, f is T-weakly commuting and $ffv=fv$ for $v\in C(f,T)$, then $P_M(u)\cap F(f)\cap F(T)\neq\emptyset$.*

Theorem 2.11. *Let M be a subset of a Banach space X, $u \in X$, $f : X \to X$ and $T : X \to CD(X)$. Assume that $P_M(u)$ is nonempty q-starshaped and weakly compact, $f(P_M(u)) = P_M(u)$, T is f-nonexpansive on $P_M(u)$, $P_M(u)$ is T-invariant, and that one of the following two conditions is satisfied*:

(a) $(f - T)$ *is demiclosed at* 0;
(b) f *is weakly continuous, T is compact-valued and X satisfies Opial's condition.*

Then $P_M(u) \cap C(f,T) \neq \emptyset$. If, in addition, f is T-weakly commuting and $ffv = fv$ for $v \in C(f,T)$, then $P_M(u) \cap F(f) \cap F(T) \neq \emptyset$.

Proof. Since $T(P_M(u)) \subset P_M(u)$, it follows that $T : P_M(u) \to CD(P_M(u))$. The result now follows from Theorem 2.7. □

Theorem 2.12. *Let M be subset of a normed space X, $f : X \to X$ and $T : X \to CD(X)$ such that $fu \in Tu = \{u\}$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Assume that $P_M(u)$ is nonempty q-starshaped and complete, $f(P_M(u)) = P_M(u)$, T is f-nonexpansive on $P_M(u) \cup \{u\}$ and that one of the following conditions holds*:

(a) $(f - T)(P_M(u))$ *is closed*;
(b) f *and T satisfy condition* (A^0).

Then $P_M(u) \cap C(f,T) \neq \emptyset$. If, in addition, f is T-weakly commuting and $ffv = fv$ for $v \in C(f,T)$, then $P_M(u) \cap F(f) \cap F(T) \neq \emptyset$.

Proof. Let $x \in P_M(u)$. Then $fx \in P_M(u)$ since $f(P_M(u)) = P_M(u)$. It follows from the definition of $P_M(u)$ that $x \in \partial M \cap M$ and since $T(\partial M \cap M) \subset M$, we have $Tx \subset M$. Let $z \in Tx$. Then

$$d(z,u) \leqslant H(Tx,Tu) \leqslant d(fx,fu) = d(fx,u) = \operatorname{dist}(u,M).$$

Now $z \in M$ and $fx \in P_M(u)$ imply that $z \in P_M(u)$. Thus $Tx \subset P_M(u)$. The result now follows from Theorem 2.9. □

The following contains, as a special case, Theorem 3 of Latif and Bano [10] and Theorem 7 of Jungck and Sessa [6].

Theorem 2.13. *Let M be subset of a normed space X, $f : X \to X$ and $T : X \to CD(X)$ such that $fu \in Tu = \{u\}$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Assume that $P_M(u)$ is nonempty q-starshaped and weakly compact, $f(P_M(u)) = P_M(u)$, T is f-nonexpansive on $P_M(u) \cup \{u\}$ and that one of the following two conditions is satisfied*:

(a) $(f - T)$ *is demiclosed at* 0;
(b) f *is weakly continuous, T is compact-valued and X satisfies Opial's condition.*

Then $P_M(u) \cap C(f,T) \neq \emptyset$. If, in addition, f is T-weakly commuting and $ffv = fv$ for $v \in C(f,T)$, then $P_M(u) \cap F(f) \cap F(T) \neq \emptyset$.

Proof. As in the proof of Theorem 2.12, $P_M(u)$ is T-invariant. The result now follows from Theorem 2.11. □

3. Random coincidence point results

The following result extends and improves Theorem 3.2 in [15].

Theorem 3.1. *Let M be a nonempty separable weakly compact q-starshaped subset of a Banach space X, and $f : \Omega \times M \to M$ a continuous and weakly continuous random operator with $f(\omega, M) = M$ for each $\omega \in \Omega$. Assume that $T : \Omega \times M \to CB(M)$ is an f-nonexpansive random operator. Suppose that one of the following two conditions is satisfied*:

(a) $(f - T)(\omega, .)$ *is demiclosed at* 0 *for each* $\omega \in \Omega$;
(b) $T(\omega, .)$ *is compact-valued for each* $\omega \in \Omega$, *and* X *satisfies Opial's condition.*

Then f and T have a random coincidence point.

Proof. By Theorem 2.2, in both of the cases, f and T have a deterministic coincidence point. The result now follows from Theorem 1.3. □

Corollary 3.2. *Let M be a nonempty separable weakly compact q-starshaped subset of a Banach space X, and $T : \Omega \times M \to CB(M)$ a nonexpansive random operator. Suppose that one of the following two conditions is satisfied*:

(a) $(I - T)(\omega, .)$ *is demiclosed at* 0 *for each* $\omega \in \Omega$;
(b) $T(\omega, .)$ *is compact-valued for each* $\omega \in \Omega$ *and* X *satisfies Opial's condition.*

Then T has a random fixed point.

Remark 3.3. Theorem 3.1 generalizes Theorem 3.4 of Tan and Yaun [17] and Theorem 1 of Xu [18].

The following result generalizes and improves Theorem 3.18 of Shahzad [13].

Theorem 3.4. *Let M be a nonempty separable closed and q-starshaped subset of a Banach space X, and $f : \Omega \times M \to M$ a continuous random operator such that $f(\omega, M) = M$ for each $\omega \in \Omega$. Assume that $T : \Omega \times M \to CD(M)$ is an f-nonexpansive random operator, f and T satisfy condition (A^0) and $T(\omega, M)$ is bounded for each $\omega \in \Omega$. Then f and T have a random coincidence point.*

Proof. By Theorem 2.4, f and T have a deterministic coincidence point. Hence, Theorem 1.4 further implies that f and T have a random coincidence point. □

The following extends and improves Theorem 3.17 [13].

Corollary 3.5. *Let M be a nonempty compact and q-starshaped subset of a normed space X, and $f : \Omega \times M \to M$ a continuous random operator such that $f(\omega, M) = M$ for each $\omega \in \Omega$. Assume that $T : \Omega \times M \to CD(M)$ is an f-nonexpansive random operator. Then f and T have a random coincidence point.*

If $f = I$, the identity map, then we get the following extensions of Corollary 3.19 [13].

Corollary 3.6. *Let M be a nonempty separable closed and q-starshaped subset of a Banach space X, and $T:\Omega\times M\to CD(M)$ a nonexpansive random operator such that $T(\omega,M)$ is bounded for each $\omega\in\Omega$. If T satisfies condition (A). Then T has a random fixed point.*

Theorem 3.7. *Suppose that X, M, f, T and q satisfy the assumptions of Theorem* 3.1 (*or Theorem* 3.4). *Moreover, if f is T-weakly commuting random operator and for any $v\in M$ and $\omega\in\Omega$, $f(\omega,f(\omega,v))=f(\omega,v)$ whenever $f(\omega,v)\in T(\omega,v)$, then f and T have a common random fixed point.*

Proof. By Theorem 3.1 (or Theorem 3.4), f and T have a random coincidence point $\psi:\Omega\to M$, i.e., $f(\omega,\psi(\omega))\in T(\omega,\psi(\omega))$ for each $\omega\in\Omega$. Let $\xi(\omega)=f(\omega,\psi(\omega))$ for $\omega\in\Omega$. Then $\xi:\Omega\to M$ is measurable. Now fix $\omega\in\Omega$. Since f is T-weakly commuting, we have $\xi(\omega)=f(\omega,\psi(\omega))=f(\omega,\xi(\omega))=f(\omega,f(\omega,\psi(\omega)))\in T(\omega,f(\omega,\psi(\omega)))=T(\omega,\xi(\omega))$. Hence ξ is a common random fixed point of f and T. □

The following extends and complements the results of Beg and Shahzad [1–3].

Theorem 3.8. *Let M be subset of a Banach space X and let $f:\Omega\times X\to X$ and $T:\Omega\times X\to CB(X)$ be random operators such that for each $\omega\in\Omega$, $u=f(\omega,u)$ and $T(\omega,u)=\{u\}$ for some $u\in X$ and $T(\omega,\partial M\cap M)\subset M$. Suppose that $P_M(u)$ is nonempty q-starshaped and that for each $\omega\in\Omega$, $T(\omega,.)$ is $f(\omega,.)$-nonexpansive on $P_M(u)\cup\{u\}$, $f(\omega,.)$ is continuous on $P_M(u)$ and $f(\omega,P_M(u))=P_M(u)$. Assume that one of the following conditions is satisfied*:

(a) *$P_M(u)$ is separable weakly compact, f is weakly continuous and $(f-T)(\omega,.)$ is demiclosed at* 0 *for each $\omega\in\Omega$*;
(b) *$T(\omega,.)$ is compact-valued on $P_M(u)$ for each $\omega\in\Omega$, $P_M(u)$ is separable weakly compact, f is weakly continuous and X satisfies Opial's condition*;
(c) *$P_M(u)$ is separable closed, and f and T satisfy condition (A^0)*;
(d) *$P_M(u)$ is compact.*

Then f and T have a random coincidence point $\psi:\Omega\to P_M(u)$. If, in addition, f is T-weakly commuting and for any $v\in M$ and $\omega\in\Omega$, $f(\omega,f(\omega,v))=f(\omega,v)$ whenever $f(\omega,v)\in T(\omega,v)$, then there exists common random fixed point $\xi:\Omega\to P_M(u)$ of f and T.

Proof. Fix $\omega\in\Omega$. As in the proof of Theorem 2.12, $P_M(u)$ is $T(\omega,.)$-invariant. We therefore obtain, in each case, that f and T have a random coincidence point $\psi:\Omega\to P_M(u)$, i.e., $f(\omega,\psi(\omega))\in T(\omega,\psi(\omega))$ for each $\omega\in\Omega$ (for (a) and (b), we apply Theorem 3.1, and for (c) and (d), we use Theorem 3.4). Let $\xi(\omega)=f(\omega,\psi(\omega))$ for $\omega\in\Omega$. Then $\xi:\Omega\to P_M(u)$ is measurable. Since f is T-weakly commuting, we have $\xi(\omega)=f(\omega,\psi(\omega))=f(\omega,\xi(\omega))=f(\omega,f(\omega,\psi(\omega)))\in T(\omega,f(\omega,\psi(\omega)))=T(\omega,\xi(\omega))$ for each $\omega\in\Omega$. Hence ξ is a common random fixed point of f and T. □

4. Further remarks

(1) All results of the paper (Theorems 2.1–3.8) remain valid if starshapedness of the set M is replaced by the following assumption considered for a single-valued case in [12]:

There exists $q \in M$ and a fixed sequence $\{k_n\}$ with $0 < k_n < 1$ converging to 1 such that $(1-k_n)q + k_n Tx \subseteq M$ for each $x \in M$.

We do not consider this case here, as it is a routine exercise.

Moreover, all results of the paper, except those for Banach spaces satisfying Opial's condition, hold if f-nonexpansiveness of T is replaced by the generalized f-nonexpansive condition:

$$H(Tx, Ty) \leqslant \max\left\{\|fx - fy\|, \operatorname{dist}\big(fx, [q, Tx]\big), \operatorname{dist}\big(fy, [q, Ty]\big), \frac{\operatorname{dist}(fx, [q, Ty]) + \operatorname{dist}(fy, [q, Tx])}{2}\right\}$$

for all $x, y \in M$.

(2) Let M be a subset of a normed space X and $S, T : M \to CD(M)$ and $f, g : M \to M$. Then S and T are called *nonexpansive relative to* f and g if

$$H(Sx, Ty) \leqslant \|fx - gy\|$$

for all $x \in M$. If both S and T are single-valued, the above definition reduces to that of Jungck [7]. Following the arguments as above, and as in Jungck [7], where single-valued case was considered, it is possible to obtain coincidence and invariant approximation results for families of four maps S, T, f and g satisfying the above inequality. We leave the obvious detail to the reader.

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